

TWO CLOSED FORMS FOR THE APOSTOL-BERNOULLI POLYNOMIALS

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ABSTRACT. “In mathematics, a closed form is a mathematical expression that can be evaluated in a finite number of operations. It may contain constants, variables, four arithmetic operations, and elementary functions, but usually no limit.” In this note, we shall obtain two closed forms for the Apostol-Bernoulli polynomials.

1. INTRODUCTION

The Bernoulli numbers B_n are defined by the generating function

$$(1.1) \quad \frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = 1 - \frac{x}{2} + \sum_{n=1}^{\infty} B_{2n} \frac{x^{2n}}{(2n)!}, \quad |x| < 2\pi$$

and the Bernoulli polynomials $B_n(u)$ are defined by

$$(1.2) \quad \frac{xe^{ux}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(u) \frac{x^n}{n!}, \quad |x| < 2\pi.$$

Because the function $\frac{x}{e^x - 1} = 1 - \frac{x}{2}$ is odd in $x \in \mathbb{R}$, all of the Bernoulli numbers B_{2n+1} for $n \in \mathbb{N}$ equal to 0. It is clear that $B_0 = 1$ and $B_1 = -\frac{1}{2}$ (see [23]).

The Apostol-Bernoulli polynomials $B_n(u, z)$ are natural generalizations of the Bernoulli polynomials, they were first introduced by Apostol [2] in order to study the Lipschitz-Lerch zeta functions (also see [21, Section 5]). Their definitions are as follows,

$$(1.3) \quad \left(\frac{x}{ze^x - 1} \right) e^{ux} = \sum_{n=0}^{\infty} B_n(u, z) \frac{x^n}{n!},$$

where $|x| \leq 2\pi$ when $z = 1$; $|x| \leq |\log z|$ when $z \neq 1$ (see [18]). In particular, $B_n(z) = B_n(0, z)$ are the Apostol-Bernoulli numbers. Letting $z = 1$ in (1.3), we obtain the Bernoulli polynomials $B_n(u)$ and Bernoulli numbers B_n , respectively.

During recent years, the Apostol-Bernoulli polynomials and numbers including their applications have been widely studied by many authors (see [3, 4, 7, 9, 10, 11, 13, 14, 15, 17, 18, 20, 22, 25, 26, 27, 28] and references therein).

Ten years ago, Jeong, Kim and Son [12] proved the following explicit formula for Bernoulli numbers, which is equivalent to an old formula by Saalschütz in 1893 [24] (also see the sentence above [12, Theorem 3.1]).

Theorem 1.1 (Jeong, Kim and Son, [12, Theorem 3.1]). *For $n \geq 1$,*

$$(1.4) \quad B_n = n! \sum_{j=1}^n (-1)^j \sum_{\substack{i_1, \dots, i_n \geq 0 \\ i_1 + \dots + i_n = j \\ i_1 + 2i_2 + \dots + ni_n = n}} \binom{j}{i_1, \dots, i_n} \frac{1}{2^{i_1} 3^{i_2} \dots (n+1)^{i_n}}.$$

The Stirling numbers $S(n, k)$ of the second kind for $n \geq k \geq 1$ can be computed and generated by

$$S(n, k) = \frac{1}{k!} \sum_{\ell=1}^k (-1)^{k-\ell} \binom{k}{\ell} \ell^n$$

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and

$$\frac{(e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n, k) \frac{x^n}{n!}$$

respectively (see [6, p. 206] and [23]).

Recently, Qi and Chapman [23] generalized the above formula to get a closed form for Bernoulli polynomials, which provide an explicit formula for computing these special polynomials in terms of Stirling numbers of the second kind $S(n, k)$.

Theorem 1.2 (Qi and Chapman, [23, Theorem 1.1]). *The Bernoulli polynomials $B_n(u)$ for $n \in \mathbb{N}$ may be expressed as*

$$(1.5) \quad B_n(u) = \sum_{k=1}^n k! \sum_{r+s=k} \sum_{\ell+m=n} (-1)^m \binom{n}{\ell} \frac{\ell!}{(\ell+r)!} \frac{m!}{(m+s)!} \\ \times \left[\sum_{i=0}^r \sum_{j=0}^s (-1)^{i+j} \binom{\ell+r}{r-i} \binom{m+s}{s-j} S(\ell+i, i) S(m+j, j) \right] \\ \times u^{m+s} (1-u)^{\ell+r}.$$

Consequently, the Bernoulli numbers B_n for $n \in \mathbb{N}$ can be represented as

$$(1.6) \quad B_n = \sum_{i=1}^n (-1)^i \frac{\binom{n+1}{i+1}}{\binom{n+i}{i}} S(n+i, i).$$

Remark 1.3. Eq. (1.6) recovers Jeong, Kim and Son's formula (1.4) (see [23, Remark 4.1]).

Remark 1.4. Using an identity involving the functions $\frac{\pm 1}{e^{\pm x} - 1}$ and their derivatives, Guo and Qi [8] also established two explicit formulas for computing Euler polynomials and two-parameter Euler polynomials in terms of Stirling numbers of the second kind.

Remark 1.5. The formulas in the above theorem are named closed forms. As recalled by Qi and Chapman [23, p. 91],

“In mathematics, a closed form is a mathematical expression that can be evaluated in a finite number of operations. It may contain constants, variables, four arithmetic operations, and elementary functions, but usually no limit.”

Furthermore, they also got another form for Bernoulli polynomials.

Theorem 1.6 (Qi and Chapman, [23, Theorem 1.2]). *Under the conventions that $\binom{0}{0} = 1$ and $\binom{p}{q} = 0$ for $q > p \geq 0$, the Bernoulli polynomials $B_n(u)$ for $n \in \mathbb{N}$ may be expressed as*

$$(1.7) \quad B_n(u) = (-1)^n \left| \frac{1}{\ell+1} \binom{\ell+1}{m} \left[(1-u)^{\ell-m+1} - (-u)^{\ell-m+1} \right] \right|_{1 \leq \ell \leq n, 0 \leq m \leq n-1},$$

where $|\cdot|_{1 \leq \ell \leq n, 0 \leq m \leq n-1}$ denotes a $n \times n$ determinant. Consequently, the Bernoulli numbers B_k for $k \in \mathbb{N}$ can be represented as

$$(1.8) \quad B_n = (-1)^n \left| \frac{1}{\ell+1} \binom{\ell+1}{m} \right|_{1 \leq \ell \leq n, 0 \leq m \leq n-1}.$$

One of main tools used in [23] to get the closed forms for the Bernoulli polynomials is the following integral expression for the generating function (1.2)

$$(1.9) \quad \frac{xe^{ux}}{e^x - 1} = \left[\frac{e^{(1-u)x} - e^{-ux}}{x} \right]^{-1} = \frac{1}{\int_{-u}^{1-u} e^{xt} dt} = \frac{1}{\int_0^1 e^{x(t-u)} dt}$$

(see [23, p. 90, (1.1)]).

In this note, we show that, different with [23], if directly applying the generating functions instead of their integral expressions, then the above theorems may be generalized to other special functions. As a result, we shall get the following two closed forms

for Apostol-Bernoulli polynomials and numbers, which may be used to compute these special polynomials and numbers in a finite number of steps.

Theorem 1.7. Suppose that $z \neq 1$. The Apostol-Bernoulli polynomials $B_n(u, z)$ for $n \in \mathbb{N}$ may be expressed as

$$(1.10) \quad B_n(u, z) = n \sum_{k=0}^{n-1} \frac{(-1)^k k!}{(z-1)^{k+1}} \sum_{r+s=k} \sum_{\ell+m=n-1} (-1)^{s+m} \binom{n-1}{\ell} \\ \times z^r (1-u)^\ell u^m S(\ell, r) S(m, s).$$

Consequently, the Apostol-Bernoulli numbers $B_n(z)$ for $n \in \mathbb{N}$ can be represented as

$$(1.11) \quad B_n(z) = n \sum_{k=0}^{n-1} \frac{(-1)^k k!}{(z-1)^{k+1}} z^k S(n-1, k).$$

Remark 1.8. Suppose that $z \neq 1$. Letting $x = 0$ in the both sides of (1.3), we get $B_0(u, z) = 0$ and $B_0(z) = 0$. Since by (1.3), we have

$$\left. \frac{d}{dx} \left(\frac{x e^{ux}}{z e^x - 1} \right) \right|_{x=0} = \sum_{m=1}^{\infty} B_m(u, z) \frac{x^{m-1}}{(m-1)!} \Big|_{x=0} = B_1(u, z)$$

and

$$(1.12) \quad \frac{d}{dx} \left(\frac{x e^{ux}}{z e^x - 1} \right) = \frac{e^{ux}}{z e^x - 1} + x \frac{d}{dx} \left(\frac{e^{ux}}{z e^x - 1} \right) \\ \rightarrow \frac{1}{z-1}$$

as $x \rightarrow 0$, so $B_1(u, z) = \frac{1}{z-1}$ and $B_1(z) = \frac{1}{z-1}$.

Remark 1.9. There is another closed form for the Apostol-Bernoulli polynomials:

$$B_n(u, z) = \sum_{k=0}^n k \binom{n}{k} \sum_{j=0}^{k-1} (-1)^j z^j (z-1)^{-j-1} j! S(k-1, j) u^{n-k}.$$

(See [16]).

Remark 1.10. Eq. (1.11) recovers a formula by Apostol (see [2, Eq. (3.7)]), and recently, Xu and Chen [28] provided another formula for the Apostol-Bernoulli numbers as follows,

$$B_n(z) = (-1)^{n-1} n \sum_{k=1}^n \frac{(k-1)!}{(z-1)^k} S(n, k)$$

(see [28, Theorem 4.1]).

Theorem 1.11. Suppose that $z \neq 1$. Under the conventions that $\binom{0}{0} = 1$ and $\binom{p}{q} = 0$ for $q > p \geq 0$, the Apostol-Bernoulli polynomials $B_{n+1}(u, z)$ for $n \in \mathbb{N}$ may be expressed as

$$(1.13) \quad B_{n+1}(u, z) = \frac{(-1)^n (n+1)}{(z-1)^{n+1}} \left| \binom{\ell}{m} [z(1-u)^{\ell-m} - (-u)^{\ell-m}] \right|_{1 \leq \ell \leq n, 0 \leq m \leq n-1},$$

where $|\cdot|_{1 \leq \ell \leq n, 0 \leq m \leq n-1}$ denotes a $n \times n$ determinant. Consequently, the Apostol-Bernoulli numbers $B_{n+1}(z)$ for $n \in \mathbb{N}$ can be represented as

$$(1.14) \quad B_{n+1}(z) = \frac{(-1)^n (n+1)}{(z-1)^{n+1}} \left| \binom{\ell}{m} (z - \delta_{\ell m}) \right|_{1 \leq \ell \leq n, 0 \leq m \leq n-1},$$

where the Kronecker delta $\delta_{\ell m}$ is 1 if the variables are equal, and 0 otherwise.

2. BELL POLYNOMIALS

As in [23], our proofs are also based on following properties of Bell polynomials. The Bell polynomials of the second kind $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ are defined by

$$(2.1) \quad B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum \frac{n!}{\ell_1! \cdots \ell_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{\ell_1} \left(\frac{x_2}{2!}\right)^{\ell_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{\ell_{n-k+1}},$$

where the sum is taken over all sequences $\ell_1, \dots, \ell_{n-k+1}$ of non-negative integers such that

$$\ell_1 + \dots + \ell_{n-k+1} = k \quad \text{and} \quad \ell_1 + 2\ell_2 + \dots + (n-k+1)\ell_{n-k+1} = n$$

for $n \geq k \geq 0$. See [6, p. 134, Theorem A].

Lemma 2.1 ([1, Example 2.6], [23, Lemma 2.1]). *For $n \geq k \geq 0$, the Bell polynomials of the second kind $B_{n,k}$ meets*

$$(2.2) \quad B_{n,k}(x_1 + y_1, x_2 + y_2, \dots, x_{n-k+1} + y_{n-k+1}) = \sum_{r+s=k} \sum_{\ell+m=n} \binom{n}{\ell} B_{\ell,r}(x_1, x_2, \dots, x_{\ell-r+1}) B_{m,s}(y_1, y_2, \dots, y_{m-s+1}).$$

Lemma 2.2 ([6, p. 135], [23, Lemma 2.2]). *For $n \geq k \geq 0$, we have*

$$(2.3) \quad B_{n,k}(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(x_1, x_2, \dots, x_{n-k+1}),$$

where a and b are any complex numbers.

Lemma 2.3 ([6, p. 135, Theorem B, [3g]]). *For $n \geq k \geq 0$, we have*

$$(2.4) \quad B_{n,k}(\underbrace{1, 1, \dots, 1}_{n-k+1}) = S(n, k).$$

3. PROOF OF THEOREM 1.7

Set

$$m(x) = x, \quad g(x) = \frac{ze^x - 1}{e^{ux}} \quad \text{and} \quad f(y) = \frac{1}{y},$$

then we have

$$h(x) = m(x)(f \circ g)(x) = \frac{xe^{ux}}{ze^x - 1}$$

which is the generating function of the Apostol-Bernoulli polynomials (1.3). Thus by (1.3), we have

$$(3.1) \quad \left. \frac{d^{n+1}}{dx^{n+1}} h(x) \right|_{x=0} = B_{n+1}(u, z).$$

On the other hand, since

$$(3.2) \quad \begin{aligned} g(x) &= \frac{ze^x - 1}{e^{ux}} = ze^{(1-u)x} - e^{-ux} \\ &= z \sum_{m=0}^{\infty} \frac{(1-u)^m}{m!} x^m - \sum_{m=0}^{\infty} \frac{(-u)^m}{m!} x^m \\ &\rightarrow z - 1 \end{aligned}$$

and

$$\begin{aligned}
(3.3) \quad g'(x) &= z \sum_{m=1}^{\infty} \frac{(1-u)^m}{(m-1)!} x^{m-1} - \sum_{m=1}^{\infty} \frac{(-u)^m}{(m-1)!} x^{m-1} \\
&\rightarrow z(1-u) - (-u), \\
g''(x) &= z \sum_{m=2}^{\infty} \frac{(1-u)^m}{(m-2)!} x^{m-2} - \sum_{m=2}^{\infty} \frac{(-u)^m}{(m-2)!} x^{m-2} \\
&\rightarrow z(1-u)^2 - (-u)^2, \\
&\vdots \\
g^{(n-k+1)}(x) &= z \sum_{m=n-k+1}^{\infty} \frac{(1-u)^m}{(m-(n-k+1))!} x^{m-(n-k+1)} \\
&\quad - \sum_{m=n-k+1}^{\infty} \frac{(-u)^m}{(m-(n-k+1))!} x^{m-(n-k+1)} \\
&\rightarrow z(1-u)^{n-k+1} - (-u)^{n-k+1}
\end{aligned}$$

as $x \rightarrow 0$.

In terms of the Bell polynomials of the second kind $B_{n,k}$, the Faà di Bruno formula for computing higher order derivatives of composite functions is described in [6, p. 139, Theorem C] by

$$(3.4) \quad \frac{d^n}{dx^n}(f \circ g)(x) = \sum_{k=0}^n f^{(k)}(g(x)) B_{n,k} \left(g'(x), g''(x), \dots, g^{(n-k+1)}(x) \right)$$

(see also [23, p. 93, (3.1)]).

By the integral expression (1.3) and (3.3), applying the formula (3.4) to the functions

$$f(y) = \frac{1}{y} \quad \text{and} \quad y = g(x) = \frac{ze^x - 1}{e^{ux}},$$

we have

$$\begin{aligned}
(3.5) \quad \frac{d^n}{dx^n}(f \circ g)(x) &= \sum_{k=0}^n f^{(k)}(g(x)) B_{n,k} \left(g'(x), g''(x), \dots, g^{(n-k+1)}(x) \right) \\
&= \sum_{k=0}^n \frac{(-1)^k k!}{(g(x))^{k+1}} B_{n,k} \left(g'(x), g''(x), \dots, g^{(n-k+1)}(x) \right) \\
&\rightarrow \sum_{k=0}^n \frac{(-1)^k k!}{(z-1)^{k+1}} \\
&\quad \times B_{n,k} \left(z(1-u) - (-u), z(1-u)^2 - (-u)^2, \dots, z(1-u)^{n-k+1} - (-u)^{n-k+1} \right)
\end{aligned}$$

as $x \rightarrow 0$ and

$$\begin{aligned}
\left. \frac{d^n}{dx^n} (f \circ g)(x) \right|_{x=0} &= \sum_{k=0}^n \frac{(-1)^k k!}{(z-1)^{k+1}} \sum_{r+s=k} \sum_{\ell+m=n} \binom{n}{\ell} \\
&\quad \times B_{\ell,r}(z(1-u), z(1-u)^2, \dots, z(1-u)^{\ell-r+1}) \\
&\quad \times B_{m,s}(-(-u), -(-u)^2, \dots, -(-u)^{m-s+1}) \\
&\quad \text{(by (2.2))} \\
&= \sum_{k=0}^n \frac{(-1)^k k!}{(z-1)^{k+1}} \sum_{r+s=k} \sum_{\ell+m=n} \binom{n}{\ell} \\
&\quad \times z^r (1-u)^\ell B_{\ell,r}(\underbrace{1, 1, \dots, 1}_{\ell-r+1}) \\
&\quad \times (-1)^s (-u)^m B_{m,s}(\underbrace{1, 1, \dots, 1}_{m-s+1}) \\
&\quad \text{(by (2.3))} \\
&= \sum_{k=0}^n \frac{(-1)^k k!}{(z-1)^{k+1}} \sum_{r+s=k} \sum_{\ell+m=n} \binom{n}{\ell} \\
&\quad \times z^r (1-u)^\ell S(\ell, r) (-1)^s (-u)^m S(m, s) \\
&\quad \text{(by (2.4)).}
\end{aligned} \tag{3.6}$$

Thus by Leibnitz's formula for the n th derivative of the product of two functions (see [30, p. 210, Example 24]), we have

$$\begin{aligned}
\frac{d^n}{dx^n} h(x) &= \frac{d^n}{dx^n} m(x)(f \circ g)(x) \\
&= \sum_{i=0}^n \binom{n}{i} (f \circ g)(x)^{(n-i)} m(x)^{(i)} \\
&= x \frac{d^n}{dx^n} (f \circ g)(x) + n \frac{d^{n-1}}{dx^{n-1}} (f \circ g)(x) \\
&\quad \text{(since } m(x) = x \text{)} \\
&\rightarrow n \sum_{k=0}^{n-1} \frac{(-1)^k k!}{(z-1)^{k+1}} \sum_{r+s=k} \sum_{\ell+m=n-1} \binom{n-1}{\ell} \\
&\quad \times z^r (1-u)^\ell S(\ell, r) (-1)^s (-u)^m S(m, s) \\
&\quad \text{(by (3.6))}
\end{aligned} \tag{3.7}$$

as $x \rightarrow 0$. Then by comparing (3.1) with (3.7), we get our result.

Finally, letting $u = 0$ in (1.10), we obtain (1.11).

4. PROOF OF THEOREM 1.11

Let $\mu = \mu(x)$ and $\nu = \nu(x) \neq 0$ be differentiable functions. Set

$$\frac{d^n}{dx^n} \left(\frac{\mu}{\nu} \right) = \frac{(-1)^n w_n}{\nu^{n+1}}$$

at every point $\nu(x) \neq 0$. By [5, p. 40], we have

$$(4.1) \quad w_n = \begin{vmatrix} \mu & \nu & 0 & 0 & \cdots & 0 \\ \mu' & \nu' & \nu & 0 & \cdots & 0 \\ \mu'' & \nu'' & 2\nu' & \nu & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu^{(n-1)} & \nu^{(n-1)} & \binom{n-1}{1}\nu^{(n-2)} & \binom{n-1}{2}\nu^{(n-3)} & \cdots & \nu \\ \mu^{(n)} & \nu^{(n)} & \binom{n}{1}\nu^{(n-1)} & \binom{n}{2}\nu^{(n-2)} & \cdots & \binom{n}{n-1}\nu' \end{vmatrix}.$$

As in [23, p. 94–95, the first proof of Theorem 1.2], we may reformulate the formula (4.1) as

$$(4.2) \quad \frac{d^n}{dx^n} \left(\frac{\mu}{\nu} \right) = \frac{(-1)^n}{\nu^{n+1}} |A_{(n+1) \times 1} \quad B_{(n+1) \times n}|_{(n+1) \times (n+1)},$$

where the matrices

$$A_{(n+1) \times 1} = (a_{\ell,1})_{0 \leq \ell \leq n} = \begin{pmatrix} \mu \\ \mu' \\ \mu'' \\ \vdots \\ \mu^{(n-1)} \\ \mu^{(n)} \end{pmatrix}$$

and

$$B_{(n+1) \times n} = (b_{\ell,m})_{0 \leq \ell \leq n, 0 \leq m \leq n-1} \\ = \begin{pmatrix} \nu & 0 & 0 & \cdots & 0 \\ \nu' & \nu & 0 & \cdots & 0 \\ \nu'' & 2\nu' & \nu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \nu^{(n-1)} & \binom{n-1}{1}\nu^{(n-2)} & \binom{n-1}{2}\nu^{(n-3)} & \cdots & \nu \\ \nu^{(n)} & \binom{n}{1}\nu^{(n-1)} & \binom{n}{2}\nu^{(n-2)} & \cdots & \binom{n}{n-1}\nu' \end{pmatrix}$$

satisfy

$$a_{\ell,1} = \mu^{(\ell)}(x) \quad \text{and} \quad b_{\ell,m} = \binom{\ell}{m} \nu^{(\ell-m)}(x)$$

under the conventions that $\nu^{(0)}(x) = \nu(x)$ and that $\binom{\ell}{m} = 0$ and $\nu^{(\ell-m)}(x) \equiv 0$ for $\ell < m$ (see [23, (3.2) and (3.3)]). If we let

$$(4.3) \quad \mu(x) = 1, \quad \nu(x) = \frac{ze^x - 1}{e^{ux}},$$

then in this case, the $a_{\ell,1}$ becomes

$$(4.4) \quad a_{0,1} = 1, \quad a_{\ell,1} = 0 \quad \text{for } \ell \geq 1$$

and the $b_{\ell,m}$ becomes

$$(4.5) \quad \begin{aligned} \binom{\ell}{m} \nu^{(\ell-m)}(x) &= \binom{\ell}{m} \frac{d^{\ell-m}}{dx^{\ell-m}} \left(\frac{ze^x - 1}{e^{ux}} \right) \\ &= \binom{\ell}{m} \frac{d^{\ell-m}}{dx^{\ell-m}} \left(ze^{(1-u)x} - e^{-ux} \right) \\ &= \binom{\ell}{m} \frac{d^{\ell-m}}{dx^{\ell-m}} \left(z \sum_{k=0}^{\infty} \frac{(1-u)^k x^k}{k!} - \sum_{k=0}^{\infty} \frac{(-u)^k x^k}{k!} \right) \\ &= \binom{\ell}{m} \left(z \sum_{k=\ell-m}^{\infty} \frac{(1-u)^k x^{k-(\ell-m)}}{(k-(\ell-m))!} - \sum_{k=\ell-m}^{\infty} \frac{(-u)^k x^{k-(\ell-m)}}{(k-(\ell-m))!} \right) \\ &\rightarrow \binom{\ell}{m} [z(1-u)^{\ell-m} - (-u)^{\ell-m}], \quad x \rightarrow 0 \end{aligned}$$

for $0 \leq \ell \leq n$ and $0 \leq m \leq n-1$ with $\ell \geq m$. Thus, as in [23, p. 95, the first proof of Theorem 1.2], by (4.2), (4.4) and (4.5), we have

$$\begin{aligned}
 (4.6) \quad \frac{d^n}{dx^n} \left(\frac{\mu}{\nu} \right) &= \frac{(-1)^n}{\nu^{n+1}} \begin{vmatrix} 1 & b_{0,0} & 0 & 0 & \cdots & 0 \\ 0 & b_{1,0} & b_{1,1} & 0 & \cdots & 0 \\ 0 & b_{2,0} & b_{2,1} & b_{2,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & b_{n-1,0} & b_{n-1,1} & b_{n-1,2} & \cdots & b_{n-1,n-1} \\ 0 & b_{n,0} & b_{n,1} & b_{n,2} & \cdots & b_{n,n-1} \end{vmatrix} \\
 &= \frac{(-1)^n}{\nu^{n+1}} |b_{\ell,m}|_{1 \leq \ell \leq n, 0 \leq m \leq n-1} \\
 &\rightarrow \frac{(-1)^n}{(z-1)^{n+1}} \left| \binom{\ell}{m} [z(1-u)^{\ell-m} - (-u)^{\ell-m}] \right|_{1 \leq \ell \leq n, 0 \leq m \leq n-1}
 \end{aligned}$$

as $x \rightarrow 0$. Then again by Leibnitz's formula for the n -th derivative of the product of two functions (see [30, p. 210, Example 24]), we have

$$\begin{aligned}
 (4.7) \quad \frac{d^{n+1}}{dx^{n+1}} \left(\frac{xe^{ux}}{ze^x - 1} \right) &= \frac{d^{n+1}}{dx^{n+1}} \left(x \left(\frac{\mu}{\nu} \right) \right) \\
 &= \sum_{i=0}^{n+1} \binom{n+1}{i} \left(\frac{\mu}{\nu} \right)^{(n+1-i)} x^{(i)} \\
 &= x \frac{d^{n+1}}{dx^{n+1}} \left(\frac{\mu}{\nu} \right) + (n+1) \frac{d^n}{dx^n} \left(\frac{\mu}{\nu} \right) \\
 &\rightarrow (n+1) \frac{d^n}{dx^n} \left(\frac{\mu}{\nu} \right) \\
 &= \frac{(-1)^n (n+1)}{(z-1)^{n+1}} \left| \binom{\ell}{m} [z(1-u)^{\ell-m} - (-u)^{\ell-m}] \right|_{1 \leq \ell \leq n, 0 \leq m \leq n-1} \\
 &\quad (\text{by (4.6)})
 \end{aligned}$$

as $x \rightarrow 0$. Thus, by comparing (3.1) with (4.7), we obtain (1.13).

Finally, letting $u = 0$ in (1.13), we obtain (1.14).

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